VI. HAAR WAVELET AND MULTIRESOLUTION ANALYSIS

In this chapter we will look at a specific Multiresolution Analysis, which is related to the Haar wavelet introduced in Chapter 3. Recall that in Example 1 of Chapter 5, we defined a sequence of subspaces V_n of $L^2(\mathbb{R})$. In fact, for any $n \in \mathbb{Z}$, we defined a subspace V_n of $L^2(\mathbb{R})$ such that $V_n = \{f \in L^2(\mathbb{R}) \big|$for any $j \in \mathbb{Z}, f|_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$ is constant $\}$.

We discussed some of their properties in Lemma 1 of Chapter 5. By definition of Multiresolution Analysis given in Chapter 5, we see that the sequence of subspaces defined above is indeed a Multiresolution Analysis.

For such a sequence $...V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$, we can define the orthogonal complement of V_0 in V_1 and call it W_0. Namely, $W_0 = V_1 \ominus V_0$. Equivalently, $V_1 = V_0 \oplus W_0$. In general, since by Lemma 1of Chapter 5, for any $n \in \mathbb{Z}$, we have $V_n \subset V_{n+1}$, so we can define the orthogonal complement of V_n in V_{n+1}, we call it W_n. Namely, $W_n = V_{n+1} \ominus V_n$. Equivalently, $V_{n+1} = V_n \oplus W_n$.

We want to investigate the orthogonal projections of any function in $L^2(\mathbb{R})$ onto subspaces V_n and W_n. To this end, we need to find for each of these subspaces a complete orthonormal system. So far, we have already obtained a complete orthonormal system for V_0. We repeat the result here.

Lemma 1. *For any $n \in \mathbb{Z}$, let $V_n = \{f \in L^2(\mathbb{R}) \big|$for any $j \in \mathbb{Z}, f|_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$ is constant $\}$. If*

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

then $\varphi(x) \in V_0$ and $\{\varphi(x-l) | l \in \mathbb{Z}\}$ is a complete orthonormal system of V_0.

Next lemma shows that Haar wavelet has something to do with a complete orthonormal system for W_0.

Lemma 2. *For any $n \in \mathbb{Z}$, let $V_n = \{f \in L^2(\mathbb{R}) \big|$for any $j \in \mathbb{Z}, f|_{[\frac{j}{2^n}, \frac{j+1}{2^n})}$ is constant $\}$ and $W_n = V_{n+1} \ominus V_n$.*

If

$$H(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & otherwise \end{cases}$$

then $H(x) \in W_0$ *and* $\{H(x-l)|l \in \mathbb{Z}\}$ *is a complete orthonormal system of* W_0*.*

Proof. We first compute to get that for each $l \in \mathbb{Z}$,

$$H_{0,l} = H(x-l) = \begin{cases} 1 & l \leq x < l + \frac{1}{2} \\ -1 & l + \frac{1}{2} \leq x < l+1 \\ 0 & otherwise \end{cases}$$

First of all, by definition of V_n, we see that $H(x-l) \in V_1$ for each $l \in \mathbb{Z}$. Secondly, for any $f \in V_0$, by definition, $f|_{[j,j+1)}$ is a constant for each $j \in \mathbb{Z}$. Let $f|_{[j,j+1)} = a_j$ for each $j \in \mathbb{Z}$, then

$$\langle f, H_{0,l} \rangle = \int_l^{l+\frac{1}{2}} a_l dx + \int_{l+\frac{1}{2}}^{l+1} (-a_l) dx = \frac{1}{2}(a_l - a_l) = 0$$

Hence $H(x-l) \in V_1 \ominus V_0 = W_0$ for each $l \in \mathbb{Z}$. Lastly, since $\{H(x-l)|l \in \mathbb{Z}\} = \{H_{0,l}|l \in \mathbb{Z}\}$ is a subset of a orthonormal system $\{H_{n,l}|l, n \in \mathbb{Z}\}$ we discussed in Chapter 3, so we only need to show that $\{H(x-l)|l \in \mathbb{Z}\} = \{H_{0,l}|l \in \mathbb{Z}\}$ is complete. By Lemma 4 in Chapter 1, we only need to show that for any $f \in V_1 \ominus V_0 = W_0$,

$$||f||_2^2 = \sum_{l \in \mathbb{Z}} |\langle f, H_{0,l} \rangle|^2.$$

Since $f \in V_1$, we can let $f|_{[\frac{j}{2}, \frac{j+1}{2})} = b_j$ for each $j \in \mathbb{Z}$. By Lamma1, $\{\varphi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of V_0, so for any $l \in \mathbb{Z}$, $\langle f, \varphi(x-l) \rangle = 0$. This means $\frac{b_0+b_1}{2} = 0$, $\frac{b_2+b_3}{2} = 0$,...,in general, for any $l \in \mathbb{Z}$, we have

$$\frac{b_{2l} + b_{2l+1}}{2} = 0.$$

On the other hand, we can compute to get that $||f||_2^2 = \sum_{l \in \mathbb{Z}} \frac{1}{2}|b_j|^2$, $\langle f, H_{0,0} \rangle = \frac{b_0-b_1}{2}$, $\langle f, H_{0,1} \rangle = \frac{b_2-b_3}{2}$,...,in general, for any $l \in \mathbb{Z}$,

$$\langle f, H_{0,l} \rangle = \frac{b_{2l} - b_{2l+1}}{2}.$$

Thus, $|\langle f, H_{0,0} \rangle|^2 = |b_0|^2$, $|\langle f, H_{0,1} \rangle|^2 = |b_2|^2$,...,in general, for any $l \in \mathbb{Z}$,

$$|\langle f, H_{0,l} \rangle|^2 = |b_{2l}|^2.$$

Hence, for any $f \in V_1 \ominus V_0 = W_0$,

$$||f||_2^2 = \sum_{l \in \mathbb{Z}} |\langle f, H_{0,l} \rangle|^2.$$

□

We still need to get complete orthonormal systems for each W_n and V_n. recall by Lemma 1 of Chapter 5, for any $n \in \mathbb{Z}$, $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$. We will see that W_n's share the same property.

Lemma 3. *Let V_n and W_n be defined above, then for any $n \in \mathbb{Z}$,*

$$f(x) \in W_n \Longleftrightarrow f(2x) \in W_{n+1}.$$

Proof. Fix any $n \in \mathbb{Z}$. We see that

$$f(x) \in W_n = V_{n+1} \ominus V_n \Longleftrightarrow f(x) \in V_{n+1}, f(x) \perp V_n.$$

By Lemma 1 of Chapter 5, certainly $f(x) \in V_{n+1} \Longleftrightarrow f(2x) \in V_{n+2}$. We are going to show that $f(x) \perp V_n \Longleftrightarrow f(2x) \perp V_{n+1}$. Indeed, note that for any $h \in L^2(\mathbb{R})$, $h(x) \in V_n \Longleftrightarrow h(2x) \in V_{n+1}$, so any function in V_{n+1} can be written as $h(2x)$ with some $h(x) \in V_n$. Now suppose $f(x) \perp V_n$, then

$$\langle f(2x), h(2x) \rangle = \int_{-\infty}^{\infty} f(2x)\overline{h(2x)}dx = \frac{1}{2}\int_{-\infty}^{\infty} f(u)\overline{h(u)}du = \frac{1}{2}\langle f, h \rangle = 0.$$

So $f(2x) \perp V_{n+1}$. The other direction is similar. Hence

$$f(x) \in W_n \Longleftrightarrow f(2x) \in V_{n+2}, f(2x) \perp V_{n+1} \Longleftrightarrow f(2x) \in W_{n+1} = V_{n+2} \ominus V_{n+1}.$$

□

If some subspaces in $L^2(\mathbb{R})$ have the above property, then their complete orthonormal systems are related in the following way.

Lemma 4. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be subspaces of $L^2(\mathbb{R})$. If $f(x) \in \mathcal{K}_1 \Longleftrightarrow f(2x) \in \mathcal{K}_2$, then the following are equivalent:*

(a) $\{f_n(x)\}_{n \in \mathbb{Z}}$ is a complete orthonormal system of $\mathcal{K}_1$.

(b)$\{\sqrt{2} f_n(2x)\}_{n \in \mathbb{Z}}$ is a complete orthonormal system of $\mathcal{K}_2$.

Proof. $(a) \Longrightarrow (b)$ From the fact that $\{f_n(x)\}_{n \in \mathbb{Z}}$ is an orthonormal system, it can be checked that $\{\sqrt{2} f_n(2x)\}_{n \in \mathbb{Z}}$ is also an orthonormal system. Furthermore, The fact that $\{f_n(x)\}_{n\in\mathbb{Z}} \subset \mathcal{K}_1$ clearly implies that $\{\sqrt{2}f_n(2x)\}_{n\in\mathbb{Z}} \subset \mathcal{K}_2$. We will show that the orthonormal system $\{\sqrt{2}f_n(2x)\}_{n\in\mathbb{Z}}$ is complete. By Lemma 4 of Chapter I, it is equivalent to show that for any $h \in \mathcal{K}_2$, we have

$$||h||_2^2 = \sum_{n\in\mathbb{Z}} |\langle h, \sqrt{2}f_n(2x)\rangle|^2.$$

Indeed, for any $h \in \mathcal{K}_2$, there is a $f \in \mathcal{K}_1$ such that $h(x) = f(2x)$. Now,

$$||h||_2^2 = \int_{-\infty}^{\infty} |h(x)|^2 dx = \int_{-\infty}^{\infty} |f(2x)|^2 dx = \frac{1}{2}\int_{-\infty}^{\infty} |f(u)|^2 du = \frac{1}{2}||f||_2^2,$$

Also

$$\langle h, \sqrt{2}f_n(2x)\rangle = \int_{-\infty}^{\infty} h(x)\overline{\sqrt{2}f_n(2x)}dx = \int_{-\infty}^{\infty} f(2x)\overline{\sqrt{2}f_n(2x)}dx$$

$$= \frac{\sqrt{2}}{2}\int_{-\infty}^{\infty} f(u)\overline{f_n(u)}du = \frac{1}{\sqrt{2}}\langle f, f_n\rangle.$$

Note that $\{f_n(x)\}_{n\in\mathbb{Z}}$ is complete in $\mathcal{K}_1$, so for such $f \in \mathcal{K}_1$, we have

$$||f||_2^2 = \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2.$$

Now we see that

$$||h||_2^2 = \frac{1}{2}||f||_2^2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 = \sum_{n\in\mathbb{Z}} |\langle h, \sqrt{2}f_n(2x)\rangle|^2.$$

$(b) \Longrightarrow (a)$ can be similarly proved. We omit the details. □

Now we are ready to find an complete orthonormal system for each subspace V_n and W_n.

Proposition 1. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a sequence of subspaces of $L^2(\mathbb{R})$ such that $V_n = \{f \in L^2(\mathbb{R})\big|$for any $j \in \mathbb{Z}, f|_{[\frac{j}{2^n},\frac{j+1}{2^n})}$ is constant $\}$. Let $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Let H be the Haar function and*

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases}$$

then for for each $n \in \mathbb{Z}$,

(a)$\{\sqrt{2^n}\varphi(2^n x - l) | l \in \mathbb{Z}\}$ is a complete orthonormal system of V_n.

(b)$\{\sqrt{2^n}H(2^n x - l) | l \in \mathbb{Z}\}$ is a complete orthonormal system of W_n.

It should be clear that Proposition 1 is an immediate consequence of the Lemmas and facts developed in this chapter. There is nothing much to prove. Finally, we have

Theorem 1. *Let* $\{V_n\}_{n\in\mathbb{Z}}$ *be a sequence of subspaces of* $L^2(\mathbb{R})$ *such that* $V_n = \{f \in L^2(\mathbb{R}) \big| \text{for any } j \in \mathbb{Z}, f|_{[\frac{j}{2^n},\frac{j+1}{2^n})} \text{ is constant }\}$. *Let* $W_n = V_{n+1} \ominus V_n$ *for each* $n \in \mathbb{Z}$. *Let* H *be the Haar function and*

$$\varphi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & otherwise \end{cases}$$

Let $H_{j,l}(x) = 2^{\frac{j}{2}} H(2^j x - l)$ *and* $\varphi_{j,l}(x) = 2^{\frac{j}{2}}\varphi(2^j x - l)$ *for each* $j, l \in \mathbb{Z}$. *Then for any* $h \in L^2(\mathbb{R})$ *and for each* $j \in \mathbb{Z}$,

$$P_{V_{j+1}}h = P_{V_j}h + P_{W_j}h.$$

Specifically,

$$\sum_{l\in\mathbb{Z}} \langle h, \varphi_{j+1,l}\rangle \varphi_{j+1,l} = \sum_{l\in\mathbb{Z}} \langle h, \varphi_{j,l}\rangle \varphi_{j,l} + \sum_{l\in\mathbb{Z}} \langle h, H_{j,l}\rangle \varphi_{j,l}.$$

Proof. Note that $V_{j+1} = V_j \oplus W_j$ for any $j \in \mathbb{Z}$, so for any $h \in L^2(\mathbb{R})$, we can consider the orthogonal projection of $P_{V_{j+1}}h \in V_{j+1}$ onto subspaces V_j and W_j. Clearly by Proposition 4 of Chapter 5, we have

$$P_{V_{j+1}}h = P_{V_j}(P_{V_{j+1}}h) + P_{W_j}(P_{V_{j+1}}h) = P_{V_j}h + P_{W_j}h.$$

Now use the complete orthonormal systems of V_n's and W_n's obtained in Proposition 1 above and Proposition 3 of Chapter 5, we can easily obtain the explicit form of $P_{V_{j+1}}h = P_{V_j}h + P_{W_j}h$. $\square$

Note that

$$\varphi_{j+1,k} = \begin{cases} 2^{\frac{1}{2^{j+1}}} & x \in [\frac{k}{2^{j+1}}, \frac{k+1}{2^{j+1}}) \\ 0 & otherwise \end{cases}$$

$$\varphi_{j,l} = \begin{cases} 2^{\frac{1}{2^j}} & x \in [\frac{2l}{2^{j+1}}, \frac{2l+2}{2^{j+1}}) \\ 0 & otherwise \end{cases}$$

and

$$H_{j,l} = \begin{cases} 2^{\frac{1}{2^j}} & x \in [\frac{2l}{2^{j+1}}, \frac{2l+1}{2^{j+1}}) \\ -2^{\frac{1}{2^j}} & x \in [\frac{2l+1}{2^{j+1}}, \frac{2l+2}{2^{j+1}}) \\ 0 & otherwise \end{cases}$$

So

$$H_{j,l} = \frac{\varphi_{j+1,2l} - \varphi_{j+1,2l+1}}{\sqrt{2}}.$$

$$\varphi_{j,l} = \frac{\varphi_{j+1,2l} + \varphi_{j+1,2l+1}}{\sqrt{2}}.$$

Thus

$$\langle h, H_{j,l} \rangle = \frac{\langle h, \varphi_{j+1,2l} \rangle - \langle h, \varphi_{j+1,2l+1} \rangle}{\sqrt{2}}.$$

$$\langle h, \phi_{j,l} \rangle = \frac{\langle h, \varphi_{j+1,2l} \rangle + \langle h, \varphi_{j+1,2l+1} \rangle}{\sqrt{2}}.$$

On the other hand, we also have

$$\langle h, \varphi_{j+1,2l} \rangle = \frac{\langle h, \varphi_{j,l} \rangle + \langle h, H_{j,l} \rangle}{\sqrt{2}},$$

$$\langle h, \varphi_{j+1,2l+1} \rangle = \frac{\langle h, \varphi_{j,l} \rangle - \langle h, H_{j,l} \rangle}{\sqrt{2}}.$$

Finally, we obtain these so called **Haar Decomposition and Reconstruction Formulas**. These formulas allow one to efficiently find the orthogonal projections of functions onto subspaces V_n and W_n. Once we take the orthogonal projection of any function f onto certain subspace V_j (for convenience, let us denote $P_{V_j} f$ as f_j), then by using Decomposition formulas, it is very easy to further find its orthogonal projection onto V_{j-1} and W_{j-1} . Denote $P_{W_j} f$ as W_j, then $f_j = f_{j-1} + w_{j-1}$. And this process can be repeated so that we can find orthogonal projections of f_{j-1} onto V_{j-2} and W_{j-2}, hence $f_j = f_{j-2} + w_{j-1} + w_{j-1}$. Then this process can be repeated once again to further decompose the original function. On the other hand , if we have the orthogonal projections of a function onto various V_n and W_n, using Reconstruction formulas we can recover the function very fast. This actually is one of the reasons why wavelets are favored by people working on signal processing. We will explain in class how this works using some signal processing example.